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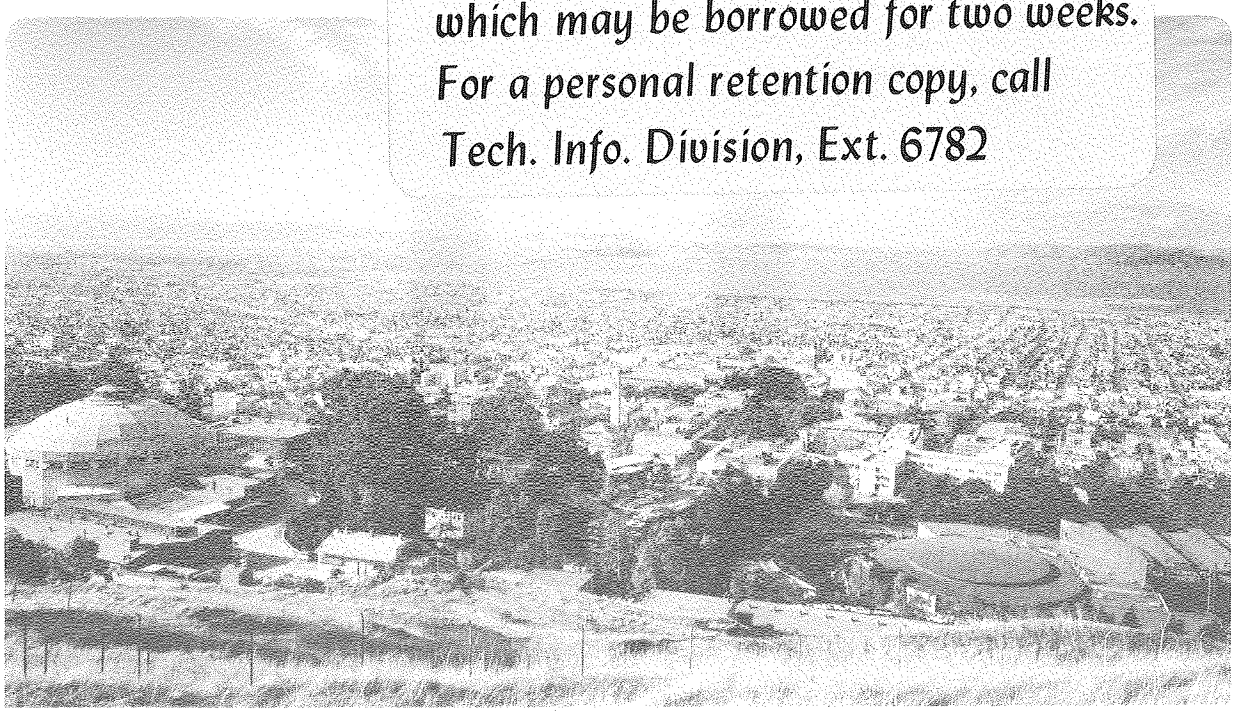

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SUMMARY

This paper develops an integral transient seepage equation that includes source terms for both fluid mass generation and undrained pore-pressure generation, and their dissipation. The equation should lead to increased accuracy for solutions involving integration over finite periods of time.

ABSTRACT FOR INFORMATION RETRIEVAL

The seepage equation is subject to two types of source terms. The first involves fluid mass generation and the resultant change in fluid potential. The second involves pore-pressure generation due to undrained response of the system to boundary load changes. Pore-pressure generation occurs at constant mass content. Since the seepage equation describes the drained problem, the inclusion of pore-pressure generation in it poses a conceptual difficulty. Also, the potential generated by either source will begin dissipating immediately after generation. This dissipation is seldom included in the statement of the seepage equation. While the omission may be acceptable in a differential equation, improved accuracy may demand inclusion of the dissipation term in the statement of the seepage problem when one wishes to obtain a solution through integration over finite intervals of time. Accordingly, a comprehensive equation governing transient seepage is developed that includes the two types of sources and their dissipation. The implications are examined.

(KEY WORDS: Seepage Equation; Pore Pressure Generation; Sources; Source Terms)

Introduction

In the equation of transient seepage the most frequently encountered source term is that which denotes the injection or withdrawal of fluid. Inasmuch as the equation of seepage is one of mass conservation, there is little difficulty in incorporating this type of source in the seepage equation. A second variety of source term, not frequently encountered but still of considerable practical importance, relates to the generation of pore pressure due to changes in the external loads acting on the porous medium. Since there is no addition or removal of mass in this case, the incorporation of this type of source needs special attention. We will term the injection/withdrawal as a source of the first kind while the pore pressure generation shall be referred to as a source of the second kind.

Furthermore, it is traditional to include the source term explicitly as a known quantity in the partial differential equation describing transient seepage. Such a treatment of the source term is adequate when one seeks to solve the problem analytically for the hydraulic head, which varies continuously in space and time. Unlike analytic solution techniques, the numerical method of solving the transient seepage equation is of a more general nature. The numerical solution is basically one of studying the overall system behavior (including, among a host of other quantities, the evolution of hydraulic head) through the implementation of the mass conservation law over finite intervals in space and in time. For purposes of using the numerical solutions, therefore, it is desirable to include the source term in the governing equation in a more systematic and detailed fashion than has hitherto been done. For example, when one seeks to numerically solve the seepage problem over finite time intervals, it is pertinent to include within the mathematical description the dissipation of fluid pressure concomitant with its generation. Pore pressure generation, in particular, is instantaneously accompanied by stepwise changes in skeletal stresses, pore volume and fluid density at constant content of fluid mass. These data are essential for a complete description of the physical problem as attempted by numerical techniques; for, in the general numerical approach, the time-wise variations of all these quantities are carefully kept accounted for.

The purpose of this paper is to state the equation of transient seepage in such a fashion that the statement is completely consistent, not only in terms of the time derivative of hydraulic head, but also in terms of changes in total stress, effective stress, void volume, fluid density and the stored fluid mass in an elemental volume.

The Transient Seepage Equation

A simple expression for the transient seepage equation, including sources, is as follows:

$$\dot{\psi}_{pp} + g_w/S_s + \nabla \cdot c_v \nabla \phi = \frac{D\psi}{Dt} \qquad (1)$$

where $\dot{\psi}_{pp}$ is the rate of generation of pressure head over time due to changes in external loads; g_w is the rate of fluid generation per unit volume of the material; $S_s = \gamma_w(a_v + ec_w)$, the specific storage coefficient, in which γ_w is unit weight of water, a_v is the coefficient of compressibility, e is the void ratio and c_w is the compressibility of water; c_v is the coefficient of consolidation, defined as $c_v = K(1 + e)/\gamma_w a_v$, where K is hydraulic conductivity; $\phi = z + \psi$, hydraulic head, where z is elevation; ∇ is the gradient operator; and D is the total derivative operator for a Lagrangian element. In Eq. 1, the total derivative is used with the understanding that the volume element of mass conservation has a constant volume of incompressible solids, V_s. Since Eq. 1 relates to an infinitestimal time interval, it does not include the dissipation of the pore pressures generated as a consequence of the two source terms. The numerical solution of Eq. 1 implies that its integration over discrete intervals is in space as well as in time. We shall now proceed to develop an integral form of the seepage equation that is more naturally amenable to numerical solution than Eq. 1.

Integral Form of the Transient Seepage Equation

Consider a small volume element of arbitrary shape enclosed by the closed surface Γ_ℓ. Let its total stress, effective stress, pressure head, fluid density and void ratio be respectively denoted by σ_ℓ, σ'_ℓ, ψ_ℓ, ρ_ℓ and e_ℓ. The volume of fluid contained in this element is $V_{w\ell} = V_{s\ell}e_\ell$ where $V_{s\ell}$ is the volume of solids (assumed to be incompressible). Let Γ_ℓ be divided into $m = 1,2,3,\ldots M$ segments, each having a surface area $\Delta\Gamma_{\ell m}$.

With time, the volume of water contained in the element will be changed by three causes. These changes are: a) Those due to hydraulic gradients existing at the initial time--$\Delta V_{w,ic}$; b) those due to sources of the first kind (i.e., fluid injection/withdrawal) and associated dissipation--$\Delta V_{w,iw}$; and c) those due to the dissipation of pore pressures generated by external load changes--$\Delta V_{w,pp}$. If we consider a small interval of time Δt, then:

$$\Delta V_{w,ic} = \Delta t \sum_{m=1}^{M} K\nabla(z + \psi)\cdot\vec{n}\Delta\Gamma_{\ell m} \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$\Delta V_{w,iw} = \int_0^{\Delta t} G_{w\ell}(t)\,dt - \Delta t \sum_{m=1}^{M} K\nabla\bar{\psi}_g\cdot\vec{n}\Delta\Gamma_{\ell m} \quad \ldots\ldots\ldots \quad (3)$$

where $\vec{n}$ is the unit outer normal, $G_{w\ell}$ is the time rate of water generation from element ℓ and $\bar{\psi}_g$ is the mean pressure head controlling drainage consequent to $G_{w\ell}$:

$$\bar{\psi}_g = \frac{1}{\Delta t}\int_0^{\Delta t} \psi_g(t)\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

in which

$$\psi_g(t) = \int_0^{t} \frac{G_{w\ell}(t)}{V_{s\ell}S_s}\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

and

$$\Delta V_{w,pp} = -\Delta t \sum_{m=1}^{M} K\nabla\bar{\psi}_{pp}\cdot\vec{n}\Delta\Gamma_{\ell m} \quad \ldots\ldots\ldots\ldots \quad (6)$$

where

$$\bar{\psi}_{pp} = \frac{1}{\Delta t}\int_0^{\Delta t} \psi_{pp}(t)\,dt \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

in which

$$\psi_{pp}(t) = \int_0^{t} \dot{\psi}_{pp}dt \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

Hence, the change in pressure head caused by drained loading is:

$$\Delta\psi_{\ell,\mathrm{drained}} = \frac{1}{V_{s\ell}S_{s\ell}}\left[\Delta V_{w,ic} + \Delta V_{w,iw} + \Delta V_{w,pp}\right] \quad \ldots \quad (9)$$

Thus, the drained mass-conservation equation is:

$$\Delta t \sum_{m=1}^{M} K\nabla(z+\psi)\cdot\vec{n}\Delta\Gamma_{\ell m} + \int_0^{\Delta t} G_{w\ell}(t)\,dt$$

$$- \Delta t \sum_{m=1}^{M} K\nabla\bar{\psi}_g\cdot\vec{n}\Delta\Gamma_{\ell m} - \Delta t \sum_{m=1}^{M} K\nabla\bar{\psi}_{pp}\cdot\vec{n}\Delta\Gamma_{\ell m}$$

$$= \Delta V_{w\ell} = V_{s\ell}S_s\Delta\psi_{\ell,\mathrm{drained}} \quad \ldots\ldots\ldots \quad (10)$$

where $\Delta V_{w\ell}$ is the change in water volume over element ℓ, and $V_{s,\ell}$ is the volume of solids in element ℓ. However, the total change in pressure head over Δt is equal to the sum of the $\Delta\psi_{\ell,\mathrm{drained}}$ and $\Delta\psi_{\ell,\mathrm{undrained}}$, where:

$$\Delta\psi_{\ell,\text{undrained}} = \psi_{pp} = \int_0^{\Delta t} \dot{\psi}_{pp} dt \quad \ldots\ldots\ldots\ldots \quad (11)$$

In view of Eqs. 10 and 11, we may write:

$$\int_0^{\Delta t} \dot{\psi}_{pp} dt + \frac{1}{V_{s\ell} S_s} \left[-\Delta t \sum_{m=1}^{M} K\nabla\bar{\psi}_{pp} \cdot \vec{n} \Delta\Gamma_{\ell m} + \int_0^{\Delta t} G_w(t)\, dt \right.$$

$$\left. - \Delta t \sum_{m=1}^{M} K\nabla\bar{\psi}_g \cdot \vec{n} \Delta\Gamma_{\ell m} + \sum_{m=1}^{M} K\nabla(z + \psi) \cdot \vec{n} \Delta\Gamma_{\ell m} \right]$$

$$= \Delta\psi_{\ell,\text{drained}} + \Delta\psi_{\ell,\text{undrained}} = \Delta\psi_{\ell} \quad \ldots\ldots \quad (12)$$

Letting $M \to \infty$ and $\Delta t \to 0$, Eq. 12 reduces to:

$$\dot{\psi}_{pp} + \frac{1}{V_{s\ell} S_s} \left[-\int_{\Gamma_\ell} K\nabla\psi_{pp} \cdot \vec{n} d\Gamma + G_{w\ell} - \int_{\Gamma_\ell} K\nabla\psi_g \cdot \vec{n} d\Gamma + \int_{\Gamma_\ell} K\nabla(z + \psi) \cdot \vec{n} d\Gamma \right]$$

$$= \frac{D\psi_{\ell,\text{drained}}}{Dt} + \frac{D\psi_{\ell,\text{undrained}}}{Dt} \quad \ldots\ldots \quad (13)$$

The discretized Eq. 12 and its integral form, Eq. 13, take into account both kinds of sources and the associated pore pressure dissipations over Δt.

Let us now proceed to derive the partial differential equation for transient seepage from Eq. 13. Noting that by definition the divergence of Darcy velocity $\vec{q}$ is:

$$\text{div } \vec{q} = \lim_{V \to 0} \frac{1}{V} \int_\Gamma \vec{q} \cdot \vec{n}\, d\Gamma \quad \ldots\ldots\ldots\ldots\ldots \quad (14)$$

if we let $V_s \to 0$ in Eq. 13, we get:

$$\dot{\psi}_{pp} + \frac{1}{S_s}\left[-\text{div } K\nabla\psi_{pp} + G_{w\ell} - \text{div } K\nabla\psi_g + \text{div } K\nabla(z + \psi)\right] = \frac{D\psi_\ell}{Dt} \quad . \tag{15}$$

If one were to solve Eq. 15 numerically using large Δt, then one has to retain, for accuracy, the two divergence terms relating to ψ_{pp} and ψ_g. However, one could, as a first-order approximation, ignore div $K\nabla\psi_{pp}$ and div $K\nabla\psi_g$ if one were to attempt an analytic solution of Eq. 15. In that case, Eq. 15 reduces to the familiar-looking expression,

$$\dot{\psi}_{pp} + \frac{1}{S_s}\left[G_{w\ell} + \sum_{m=1}^{M} K\nabla(z + \psi)\cdot\vec{n}\, d\Gamma\right] = \frac{D\psi_\ell}{Dt} \quad \ldots\ldots \tag{16}$$

Even in Eq. 16 the presence of the undrained term $\dot{\psi}_{pp}$ introduces a dimensionality problem. Thus, note that on the left-hand side of Eq. 16, S_s does not divide $\dot{\psi}_{pp}$. Because of the nature of the differential equation, little can be done to express $\dot{\psi}_{pp}$ in a more acceptable form. Indeed, if one were to abandon the continuum assumptions inherent in the partial differential equation (Eq. 16), then it is logically much easier to handle $\dot{\psi}_{pp}$. Thus, if we choose to define the initial condition to be that at the start of the discrete interval Δt, then one may simply add $\dot{\psi}_{pp}$ to the initial condition, rather than mixing it with the governing equation, which, in fact, deals exclusively with drained conditions of flow. Accordingly, the complete statement of the seepage problem in an integral form is as follows: For any appropriately small volume element ℓ within the flow region,

$$- \int_{\Gamma_\ell} K\vec{\nabla}\psi_{pp} \cdot \vec{n} d\Gamma + G_w - \int_{\Gamma_\ell} K\vec{\nabla}\psi_g \cdot \vec{n} d\Gamma + \int_{\Gamma_\ell} K\nabla(z + \psi) \cdot \vec{n} d\Gamma = V_{s\ell} S_s \frac{D\psi_\ell}{Dt} \quad (17)$$

Initial Condition

$$\psi_\ell(t_o) = \psi_\ell^\circ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

Auxiliary Condition 1

$$\psi_\ell(t_o + \Delta t) = \psi_\ell^\circ + \psi_{pp} + \frac{D\psi_\ell}{Dt} \Delta t \quad \ldots\ldots\ldots\ldots \quad (19)$$

Boundary Conditions

Boundary conditions are implicit in the three surface integrals on the left-hand side of Eq. 17. Thus, $\Gamma_\ell = \Gamma_{\ell,i} + \Gamma_{\ell,b}$, where $\Gamma_{\ell,i}$ is that portion of Γ_ℓ lying entirely within the flow region and $\psi_{\ell,b}$ is that portion of Γ_ℓ coinciding with the external boundary of the flow region. For $\Gamma_{\ell,b}$, part of the information needed to evaluate the integral is *a priori* known either in the form of prescribed potential conditions, prescribed flux conditions or both (e.g., seepage face, free surface).

Pore Pressure Generation and Mass Balance

A discussion of pore pressure generation cannot be complete without considering its role in mass balance. Because pore pressure is generated instantaneously in response to external loads, the volume of fluid (actually, the mass of fluid) within the element remains constant until pore pressure begins to dissipate. Yet, the properties σ_ℓ, σ'_ℓ, ψ_ℓ, ρ_ℓ, and e_ℓ all change in a stepwise fashion along with the instantaneous generation of pore pressure. Only the fluid mass content of the volume element remains continuous when pore pressure is generated. These important consequences are schematically shown in Fig. 1.

A serious limitation of a differential equation such as Eq. 15 in dealing with thick natural systems is that it states the problem purely in terms of fluid pressure. Properties such as void ratio are often strong functions of effective stress. In thick systems, for example, the same pore pressure at two different depths may be associated with different values of σ, σ', ρ, e and a_v. Therefore, a knowledge of the total stress distribution in such systems, in addition to a knowledge of the pressure head distribution, is essential if a realistic analysis is to be made. Such information is extremely difficult to incorporate into a statement of the differential equation that treats hydraulic head and pressure head as continuous functions in space. Many realistic field problems lead to such pronounced nonlinearities that the numerical approach of using spatially and temporally discontinuous functions is now widely sought after, in preference to analytic methods of solution. As shown in Eqs. 17 through 19, the governing integral equations of these discontinuous systems possess great flexibility in accommodating a variety of auxiliary statements essential for comprehensive problem statement. With regard to the two kinds of source terms already described, Auxiliary Condition 2 is needed to reinforce the seepage equation:

Auxiliary Condition 2

$$\sigma_{\ell}(t_o + \Delta t) = \sigma_{\ell}(t_o) + \gamma_w \psi_{pp}\left(1 + \frac{ec_w}{a_v}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (20)$$

Practical Implications

The implication of Eq. 20 is that the total stress for every volume element has to be continuously known as the system evolves in time. Equations 17 through 20 completely describe the transient seepage problem. Implemented faithfully, these equations will make it possible not only to follow the evolution of hydraulic head as a function of time, as is the case with the classical differential equation, but also to follow the evolution of the total and effective stresses, void ratio and fluid density as well.

Furthermore, when the system being studied is a complex one, the only logical method of validating the results is to verify whether at any given time the set of the fundamental parameters σ, σ', e and ρ is fully consistent with the law of mass conservation as the system gradually evolves with time. In addition to being axiomatically sound, such a validation is extremely desirable from an engineering viewpoint because the relationship of the computational problem to the real-life physical quantities always remains in focus.

Conclusion

In this paper, sources involving the generation of mass are termed sources of the first kind. On the other hand, pore pressures generated by external loads under constant fluid storage are referred to as sources of the second kind. Sources of the second kind lead to stepwise changes in total stress, effective stress, fluid pressure, void ratio and fluid density as a function of time. For a comprehensive analysis of the transient seepage problem, the evolution of these stepwise changes has to be followed in a systematic fashion. These changes, however, cannot easily be incorporated in the statement of the transient problem as a classical differential equation. Instead, it is shown that the problem needs to be stated as an integral equation of mass conservation for the drainage phenomenon, reinforced by statements of initial, boundary and auxiliary conditions.

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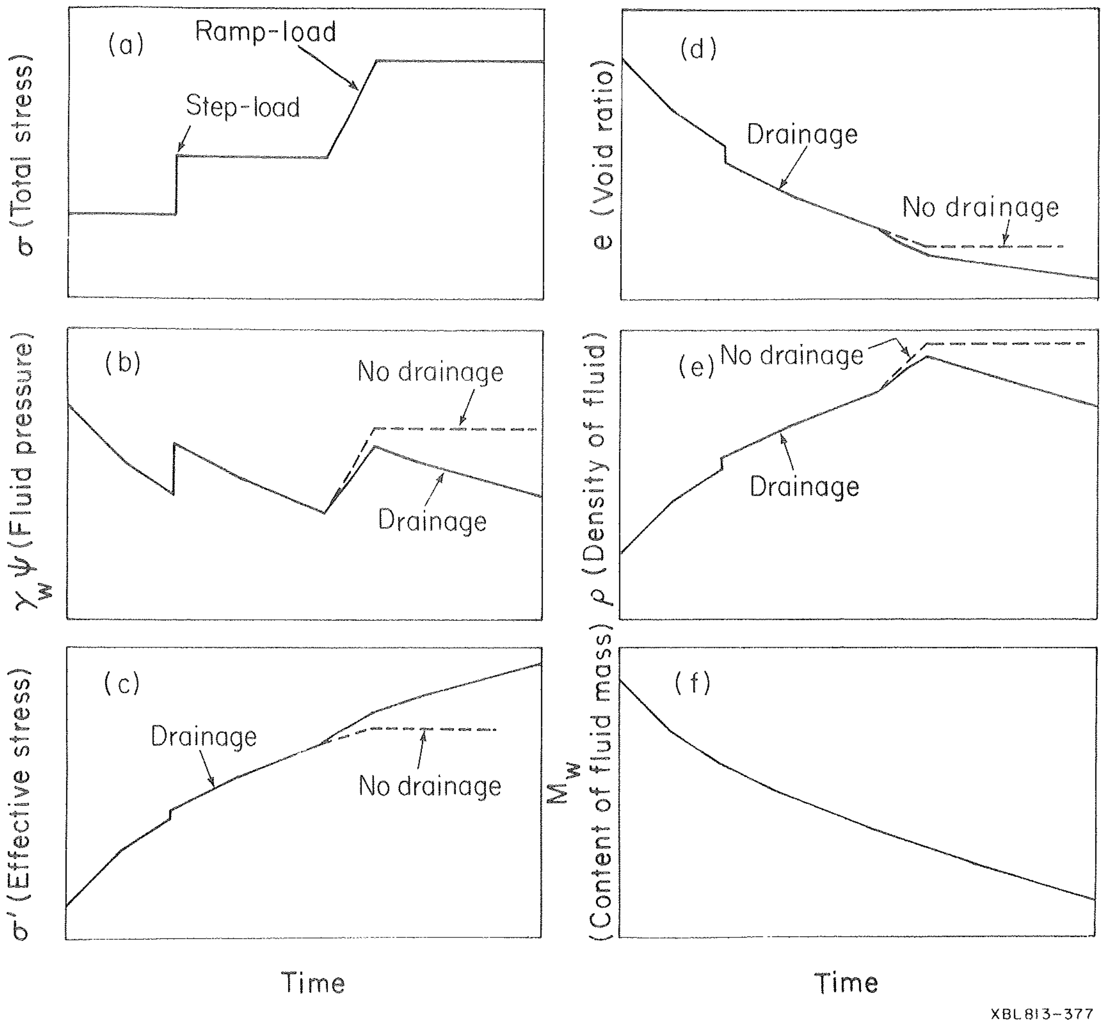
Appendix I. Notation

a_v	=	coefficient of compressibility $[LT^2/M]$
c_v	=	coefficient of consolidation $[L^2/T]$
c_w	=	compressibility of water $[LT^2/M]$
D	=	total derivative operator for a Lagrangian element
e	=	void ratio
g_w	=	rate of fluid generation per unit volume of material $[L^3/L_3T]$
$G_{w\ell}$	=	volumetric rate of generation of water from element ℓ $[L^3/T]$
K	=	hydraulic conductivity [L/T]
ℓ	=	volume element
M_w	=	content of fluid mass = ρV_w
$\vec{n}$	=	unit outer normal
$\vec{q}$	=	specific flux or Darcy velocity
S_s	=	specific storage = $\gamma_w(a_v + ec_w)$ [1/L]
t	=	time
Δt	=	interval of time [T]
V_s	=	volume of solids $[L^3]$
V_w	=	volume of water $[L^3]$
$\Delta V_{w,ic}$	=	change in water volume due to initial conditions $[L^3]$
$\Delta V_{w,iw}$	=	change in water volume due to sources of the first kind $[L^3]$
$\Delta V_{w\ell}$	=	change in water volume over element ℓ $[L^3]$
$\Delta V_{w,pp}$	=	change in water volume due to dissipation accompanying sources of the second kind

z	=	elevation head [L]
γ_w	=	unit weight of water $[M/LT^2]$
Γ_ψ	=	closed surface bounding element ℓ $[L^2]$
$\Delta\Gamma_{\ell m}$	=	mth segment of Γ_ℓ $[L^2]$
ρ	=	density of fluid $[M/L^3]$
σ	=	total stress $[M/LT^2]$
σ'	=	effective stress $[M/LT^2]$
ϕ	=	hydraulic head $= z + \psi$
ψ	=	pressure head [L]
ψ_g	=	pressure head generated over the interval $(t - t_o)$ due to sources the first kind [L]
$\bar{\psi}_g$	=	mean value of ψ_g over Δt [L]
ψ_{pp}	=	pressure head generated over the interval $(t - t_o)$ due to sources of the second kind [L]
$\bar{\psi}_{pp}$	=	mean value of ψ_{pp} over Δt [L]
$\dot{\psi}_{pp}$	=	rate of generation of ψ [L/G]
∇	=	gradient operator

FIG. 1--Simultaneous response of a saturated medium to a change in external stress and drainage. A stepwise change in external loads, as well as a ramplike change in load, is illustrated. (a) Total stress, σ; (b) fluid pressure, γ_w ; (c) effective stress, σ'; (d) void ratio, e; (e) fluid density, ρ; (f) content of fluid mass, $M_w = \rho V_w$. Note that only M_w is a fully continuous function of time.



(a)
Ramp-load
Step-load
σ (Total stress)
(b)
No drainage
Drainage
$\gamma_w \psi$ (Fluid pressure)
(c)
Drainage
No drainage
σ' (Effective stress)
Time
(d)
Drainage
No drainage
e (Void ratio)
(e)
No drainage
Drainage
ρ (Density of fluid)
(f)
M_w (Content of fluid mass)
Time
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Fig. 1